ON THE ULAM-ZAHORSKI PROBLEM

The Ulam-Zahorski Problem is the problem of establishing "interpolation links" between the classes of functions $f : [0, 1] \to \mathbb{R}$ indicated in the following diagram

$$A \to C^\infty \to \ldots \to C^n \to D^n \to \ldots \to C^1 \to D^1 \to C^0 \quad (RA),$$

or alternatively, the following diagram

$$A \to C^\infty \to \ldots \to C^n \to C^{(n)} \to \ldots \to C^1 \to C^{(1)} \to C^0 \quad (HA).$$

$C^0$ denotes the class of continuous functions, $D^n$ the $n$-times differentiable functions, $C^n$ the $n$-times continuously differentiable functions, $C^\infty$ the infinitely differentiable functions, $A$ the real analytic functions, $C^{(1)}$ the Lipshitz functions, and $C^{(n)}$ the $n$th Hölder class, consisting of the functions $f \in C^{n-1}$ whose $n-1$th derivative $f^{(n-1)} \in C^{(1)}$. The notation $\to$ can be interpreted as meaning "$\subseteq" or "$\Rightarrow". The author thinks of (RA) as the "Real Analyst's Mainstream" of function classes and (HA) as the "Harmonic Analyst's Mainstream". If $X \to \ldots \to Y$ denote two of the classes above, $X$ and $Y$ are said to be "linked" if every $f \in Y$ agrees on some uncountable set with some $g \in X$. The existence and nonexistence of such links will be denoted by

$$\ldots \to X \to \ldots \to Y \to \ldots$$

and

$$\ldots \to X \to \ldots \to Y \to \ldots \quad \boxplus,$$

respectively. Ulam asked [Scottish Book Problem 17.1] (see [4]) whether every $f \in C^0$ agrees on some uncountable set with a $g \in A$. Zahorski showed [7] in 1947 that the answer is "no", because

$$A \to C^\infty \to \ldots \quad \boxplus.$$

Zahorski then asked whether every $f \in C^0$ agrees on some uncountable set with some $g \in C^\infty$ (or $C^n$ or $D^n$). Since that time, theorems (and examples) which established more interpolation links (and failed links) have
coming. Results due to Whitney [6] (1951), Agronsky, Bruckner, Laczkovich, and Preiss [1] (1985), and Olevskii [5] (1994) have provided various parts of the solution of the problem since Zahorski's paper. The current status of the problem, as far as links and failed links in the two streams of function classes are concerned, is illustrated by the following two diagrams:

\[ ... \to C^{n+1} \to C^{(n+1)} \to C^n \to ... \to C^2 \to C^{(2)} \to C^1 \to C^{(1)} \to C^0 \quad (HA) \]

\[ ... \to C^{n+1} \to D^{n+1} \to C^n \to ... \to C^2 \to D^2 \to C^1 \to D^1 \to C^0 \quad (RA) \]

and for every \( n \geq 2 \). It is clear that, as far as the (HA) stream is concerned, the problem is completely solved, as Olevskii stated in [5] (although the \( C^n-C^{(n)} \) and \( C^n-D^n \) links were not explicitly mentioned in [5]). In [2] (pg. 523), the current author mistakenly indicated that the problem was also completely solved as far as the (RA) stream is concerned, but the diagram makes it clear that there are still a number of unanswered questions concerning the \( D^n \)-links. The author found that there are many possibilities for the the form the final complete solution of the problem for the (RA) stream might take. The most bizarre possibility includes the following:

\[ C^3 \to D^3 \to C^2 \to D^2 \to C^1 \to D^1 \to C^0 \]

However, the author is betting that the final complete solution will be one of the following two possibilities:

\[ ... \to C^{n+1} \to D^{n+1} \to C^n \to ... \to C^2 \to D^2 \to C^1 \to D^1 \to C^0 \quad (1) \]

or

\[ ... \to C^{n+1} \to D^{n+1} \to C^n \to ... \to C^2 \to D^2 \to C^1 \to D^1 \to C^0 \quad (2) \]

for every \( n \geq 2 \). The only (supposedly) "new" result announced by the au-
Author at the Symposium was the existence of an \( f \in D^1 \) such that for every \( g \in D^2 \), \( \{ x : f(x) = g(x) \} \) is of Lebesgue measure zero. However, after the Symposium ended, the author realized that his example is not new. In fact, one can get a better example (where \( f \in C^1 \)) by taking the indefinite integral of an old example (described by Jarník in 1934 [3]) of a continuous function which was almost nowhere approximately differentiable. But if one could show that \( \{ x : f(x) = g(x) \} \) were countable in the example mentioned above, this would be progress toward solution of the Ulam-Zahorski problem for the (RA) stream.

References


